A queueing system with n phases of service, vacations and retrial customers

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Abstract

A queueing system with a single server providing n phases of service in succession is considered. Every customer receives service in all phases. When a customer completes his service in the i^{th} phase he decides either to proceed to the next phase of service or to join the K_i retrial box ($i = 1, 2, ..., n-1$), from where he repeats the demand for the $(i+1)^{th}$ phase of service after a random amount of time and independently to the other customers in the system. When there are no more customers waiting in the ordinary queue (first stage), the server departs for a single vacation of an arbitrarily distributed lenght. The arrival process is assumed to be Poisson and all service times are arbitrarily distributed. For such a system, the mean number of customers in the ordinary queue and in each retrial box separately are obtained, and used to investigate numerically system performance.

Keywords: Poisson arrivals, n-phase service, retrial queues, general services, single vacation.

1 Introduction

Queueing systems in which the server provides to each customer a number of phases of heterogeneous service in succession, can be proved very useful to model computer networks, production lines and telecommunication systems where messages are processed in n stages by a single server.

Such kind of systems, with only two phases of service, have firstly discussed by Krishna and Lee [9] and Doshi [5], while more recently in a series of works (Madan [13], Choi and Kim [2], Choudhury and Madan [4], Katayama and Kobayashi [7], Ke [8]), the previous results, are extended to include systems allowing server vacations, Bernoulli feedback, N-policy, exhaustive or gated bulk service, startup times etc., but again for models with only two phases of service. Moreover in all papers mentioned above one can find important applications to computer communication, production and manufacturing systems, central processor units and multimedia communications.

Kumar, Vijayakumar and Arivudainambi [11] and Choudhury [3] are the first who imposed the concept of "retrial customers " in the two phase service models. Retrial queueing systems are characterized by the fact that an arriving customer who finds the server unavailable does not wait in a queue but instead he leaves the system, joining the so called retrial box, from where he repeats the demand for service later. Practical use of retrial queueing systems arises in telephone-switching systems and in telecommunication and computer networks. For complete surveys of past papers on such kind of models see Falin and Templeton [6] , Kulkarni and Liang [10] and Artalejo [1]. Kumar et.al. [11] considered a two phase service system where an arriving customer who finds the server unavailable joins the retrial box from where only the first customer can retry for service after an arbitrarily distributed time period while in the work of Choudhury [3] the investigated two phase model includes Bernoulli server vacations and linear retrial policy. We have to observe here that in both papers the service procedure contains only two phases of service and also there is not any ordinary queue and all "waiting " customers are placed in the retrial box.

In the work here we consider, for first time in the literature, a model with n phases of service and $n-1$ retrial boxes, $K_1, K_2, ..., K_{n-1}$ say. All arriving customers are placed, upon arrival, in an ordinary queue (first stage) to receive service. When a customer completes his first phase service then, with probability $1-p_1$, he proceeds to the second phase while, with probability p_1, he leaves the system and joins the K_1 retrial box. This procedure is repeated in each stage and so, when a customer completes his i^{th} phase service, then either, with probability $1-p_i$, he proceeds to the $(i+1)^{th}$ phase, or with probability p_i he joins the K_i retrial box. The customers in each retrial box retry, after a random amount of time and independently to each other, to find the server available and to proceed to the next phase of their service. Note here that every customer can join more than one retrial boxes during his service procedure. Moreover, when there are no more customers for service in the ordinary queue, the server departs for a single vacation (update devices, maintenance, etc.) of arbitrarily distributed length. We have to point out here that in our model, and at any time, an ordinary and $n-1$ retrial queues must be taken in to account and so the way to handle the situation becomes much more complicated.

Our system can be used to model any situation with many stages of service, where in each stage a control and a separation of the serviced units must be taken place, and if a unit satisfies some quality standards then it proceeds immediately to the next phase of service, while, if the quality of the unit is poor, then it is removed from the system and repeats its attempt to continue its service procedure later when the server is free from high quality units. As one understand a such kind of situation arise often in packet transmissions, in manufacturing systems, in central processors, in multimedia communications etc..

The article is organized as follows. A full description of the model is given in section 2. Some, very useful for the analysis results, on the customer total service cycle and server busy and vacation periods, are given in section 3, while a system states analysis is performed in section 4. In section 5 the mean number of customers in the ordinary queue and in each retrial box separately are obtained, and used to produce, in section 6, numerical results and to compare numerically system performance under various changes of the parameters.

2 The model

Consider a queueing system consisting of n phases of service and a single server who follows the customer in service when he passes from one phase to the next. Customers arrive to the system according to a *Poisson* distribution parameter λ, and are placed in a single queue (first phase) waiting to be served. When a customer finishes his service in the i^{th} phase $(i = 1, 2, ..., n)$ then either he goes to the $(i+1)^{th}$ phase with probability $1-p_i$, or he departs from the system with probability p_i and joins the K_i^{th} retrial box $(i = 1, 2, ..., n-1)$ from where he retries, independently to the other customers in the box, after an exponential time parameter μ_i, to find the server idle and to proceed to the $(i+1)^{th}$ phase of service. In case the customer chooses to depart and to join the retrial box, the server starts immediately to serve in the first phase the next customer in queue (if any). Every time the server becomes idle (no customers waiting in the ordinary queue) he departs for a single vacation U_0 which length is arbitrarily distributed with distribution function (D.F.) $B_0(x)$, probability density function (p.d.f.) $b_0(x)$ and finite mean value $\bar{b}_0$ and second moment about zero $\bar{b}_0^{(2)}$. If the server, upon returning from a vacation, finds customers waiting for service in the first stage (ordinary queue) he starts serving them immediately, while if there are no customers waiting, he remains idle awaiting the first arrival, from outside or from a retrial box, to start the service procedure again.

Let us call P_1 customers the ordinary customers who are queued up and wait to be served and P_i customers $(i = 2, 3, ..., n)$ those who joined the K_{i-1}^{th} retrial box. Note here that any customer can join, during his service procedure, a number of retrial boxes and so a P_i customer is called P_i customer as far as he continues his service procedure passing from one phase of service to the next without joining another retrial box, while if a P_i customer joins in the sequel the K_{j-1}^{th} retrial box $(j > i)$ then he becomes a P_j customer. The service time of a P_i customer in the j^{th} phase, B_{ij} say, is assumed to be arbitrarily distributed with D.F. $B_{ij}(x)$, p.d.f. $b_{ij}(x)$ and finite mean value $\bar{b}_{ij}$ and second moment about zero $\bar{b}_{ij}^{(2)}$ ($B_{ij}(x)$, $b_{ij}(x)$, $\bar{b}_{ij}, \bar{b}_{ij}^{(2)}$ do not exist of course for $j < i$). Finally all random variables defined above are assumed to be independent.

3 General Results

If a customer does not join any retrial box during his service procedure (with probability $\bar{p}_0 = \prod_{i=1}^{n-1}(1 - p_i)$) then his **total service cycle** will be $\bar{R}_0 =$

$\sum_{j=1}^{n} B_{1j}$ with LST of its p.d.f.

$$\bar{r}_0(s) = \prod_{j=1}^{n} \beta^*_{1j}(s),$$

with $\beta^*_{ij}(\cdot)$ the LST of $b_{ij}(\cdot)$. Let us suppose now that an arriving customer joins r retrial boxes ($r = 1, 2, ..., n-1$) during his service procedure, for example he joins the retrial boxes K_{m_1}, K_{m_2}, ..., K_{m_r}. Then it is clear that $m_1 = 1, 2, ..., n-r$, $m_2 = m_1+1, ..., n-r+1$, and so on, until $m_r = m_{r-1}+1, ..., n-1$. Moreover the probability of this event is

$$\bar{p}_{m_1 m_2 ... m_r} = p_{m_1} p_{m_2} \cdots p_{m_r} \prod_{\substack{i=1 \\ i \neq m_1, ..., m_r}}^{n-1} (1 - p_i),$$

while the duration of the customer's **total service cycle** in this case is (with $m_0 \equiv 0$)

$$\bar{R}_{m_1 m_2 ... m_r} = \sum_{i=1}^{r} \Big(\sum_{j=m_{i-1}+1}^{m_i} B_{m_{i-1}+1j} + \overline{V} \Big) + \sum_{j=m_r+1}^{n} B_{m_r+1j},$$

and the LST of its p.d.f.

$$\bar{r}_{m_1 m_2 ... m_r}(s) = (\overline{v}^*(s))^r \prod_{i=1}^{r+1} \prod_{j=m_{i-1}+1}^{m_i} \beta^*_{m_{i-1}+1j}(s),$$

with $m_{r+1} \equiv n$. Note here that $\overline{V}$ is the delay incurrent due to server absence (in vacations) that precedes the service of every customer emerging from a retrial box. This absence can be either of a single duration U_0, if no customers arrive from outside during the vacation U_0, or of a multiple duration, if at least one customer arrives during U_0, in which case the server has to repeat the vacation as soon as he finishes the busy period of P_1 customers and before he becomes available to the customer emerging from the retrial box. Thus the p.d.f. $\overline{v}(t)$ of $\overline{V}$ satisfies

$$\overline{v}(t) = e^{-\lambda t} b_0(t) + (1 - e^{-\lambda t}) b_0(t) * \overline{v}(t),$$

with LST

$$\overline{v}^*(s) = \frac{\beta_0^*(\lambda + s)}{1 + \beta_0^*(\lambda + s) - \beta_0^*(s)}.$$

Thus the LST of the p.d.f. of the customer's **total service cycle** $\bar{R}$ is given by

$$\bar{r}(s) = \bar{p}_0 \bar{r}_0(s) + \sum_{r=1}^{n-1} \sum_{m_1=1}^{n-r} \sum_{m_2=m_1+1}^{n-r+1} \cdots \sum_{m_r=m_{r-1}+1}^{n-1} \bar{p}_{m_1 m_2 ... m_r} \bar{r}_{m_1 m_2 ... m_r}(s),$$

and if we take derivatives above, at $s=0$, we arrive after some algebra at

$$\rho^* \equiv -\lambda \frac{d}{ds}\bar{r}(s)|_{s=0} = \lambda E(\bar{R}) = \sum_{j=1}^{n}(\rho_j + \rho_{0j}), \tag{1}$$

where

$$\begin{aligned} \rho_{0j} &= \tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} p_{j-1}, \qquad j = 2,3,...,n, \\ \rho_j &= \lambda p_{j-1}[\bar{b}_{jj} + \textstyle\sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1}(1-p_m)], \qquad j = 1,2,...,n. \end{aligned} \tag{2}$$

with $p_0 \equiv 1$, $\rho_{01} \equiv 0$. Thus ρ^* must be considered as the mean number of new customers arriving during $\bar{B}$ and so, for an ergodic system, we have to assume $\rho^* < 1$.

Let now S_j be the time interval from the epoch at which a P_j customer starts his service in the j^{th} phase until the epoch he either completes his service procedure and depart from the system or he joins another retrial box and releases the server. Let also $N_i(S_j)$ be the new P_i customers during S_j. Note here that during S_j we can have only new P_1customers (external arrivals) and/or one and only one new P_{j+1} or P_{j+2} or ...or P_n customer according to the retrial box that this specific P_j customer will join next. Define finally

$$a^{(j)}(t, k_1,\ k_{j+1}, ..., k_n)dt = P[N_i(S_j) = k_i \ \ i = 1, j+1, ..., n,\ \ t < S_j \le t + dt],$$
$$a_j(z_1, z_{j+1}, ..., z_n) =$$
$$\sum_{k_1=0}^{\infty} \sum_{k_{j+1}=0}^{1} \cdots \sum_{k_n=0}^{1} z_1^{k_1} z_{j+1}^{k_{j+1}} ... z_n^{k_n} \int_{t=0}^{\infty} a^{(j)}(t, k_1,\ k_{j+1}, ..., k_n)dt.$$

Then it is easy to understand that, for any $j = 1,2,...,n$,

$$a_j(z_1, z_{j+1}, ..., z_n) = \textstyle\sum_{m=j}^{n} p_m z_{m+1} \prod_{l=j}^{m-1}(1-p_l) \prod_{k=j}^{m} \beta_{jk}^*(\lambda - \lambda z_1), \tag{3}$$

with $p_n \equiv 1$, $z_{n+1} \equiv 1$. Moreover from relation (2)

$$\frac{d}{dz_1} a_1(z_1, 1, 1, ..., 1)|_{z_1=1} = \rho_1.$$

In general and for any x_j $j = 1,2,...,n$, let as denote, for simplicity, by $\boldsymbol{x}$ the $(1 \times n-1)$ vector $\boldsymbol{x} = (x_2, x_3, ..., x_n)$ and by $\bar{\boldsymbol{x}}$ the $(1 \times n)$ vector $\bar{\boldsymbol{x}} = (x_1, x_2, ..., x_n)$. To proceed further we need the following Lemma the proof of which is a simple application of the well known theorem of Takacs [15].

Lemma 1 *If (i) $|z_k| < 1$ for any specific $k = 2, ..., n$, and $|z_m| \le 1$ for all other $2 \le m \le n$ with $m \ne k$, or (ii) $|z_m| \le 1$, for all $2 \le m \le n$ and $\rho_1 > 1$, then the relation*

$$z_1 - a_1(z_1, z_2, ..., z_n), \tag{4}$$

has one and only one zero, $z_1 = x(\boldsymbol{z})$ say, inside the region $|z_1| < 1$. Specifically for $\boldsymbol{z} = \mathbf{1}$, $x(\mathbf{1})$ is the smallest positive real root of (4) with $x(\mathbf{1}) < 1$ if $\rho_1 > 1$ and $x(\mathbf{1}) = 1$ for $\rho_1 \le 1$.

Let now $T^{(i)}$ be the duration of a busy period of P_1 customers starting with i P_1 customers and $N_j(T^{(i)})$ be the number of new P_j customers (joining the K_{j-1}^{th} retrial box) during $T^{(i)}$. Define

$$g_{k}^{(i)}(t)dt = P[N_j(T^{(i)}) = k_j \;\; j = 2, 3, ..., n, \;\; t < T^{(i)} \leq t + dt],$$
$$G^{(i)}(s, z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} g_{k}^{(i)}(t)dt.$$

Now it is easy to see (Theorem 1 in Langaris and Katsaros [12]) that

$$G^{(i)}(0, z) = x^i(z),$$

where $x(z)$ the only zero of $z_1 - a_1(z_1, z_2, ..., z_n)$ in $|z_1| < 1$.

Let now V be the time interval from the epoch the server departs for a single vacation until the epoch he becomes idle for the first time. Denote also $N_j(V)$ the number of new P_j customers during V. If we define

$$v_k(t)dt = P[N_j(V) = k_j \;\; j = 2, 3, ..., n, \;\; t < V \leq t + dt],$$
$$v^*(s, z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} v_k(t)dt,$$

then

$$v_k(t) = e^{-\lambda t} b_0(t) \delta_{\{k=0\}} + \sum_{m=0}^{k} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} b_0(t) * g_{m}^{(i)}(t) * v_{k-m}(t),$$

where

$$\delta_{\{A\}} = \begin{cases} 1 & if \;\; A \;\; holds \\ 0 & otherwise, \end{cases}$$

and so

$$v^*(0, z) = \frac{\beta_0(\lambda)}{1 + \beta_0(\lambda) - \beta_0(\lambda - \lambda x(z))}. \tag{5}$$

Let $D^{(i)}$ the time interval from the epoch a P_i $i = 2, 3, ..., n$ retrial customer finds a position for service until the epoch that the server departs for a vacation, and denote by $N_j(D^{(i)})$ the number of the new P_j customers during $D^{(i)}$. Define finally

$$d_{k}^{(i)}(t)dt = P[N_j(D^{(i)}) = k_j \;\; j = 2, 3, ..., n, \;\; t < D^{(i)} \leq t + dt],$$
$$D^{(i)}(s, z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} d_{k}^{(i)}(t)dt,$$

then

$$d_{k}^{(i)}(t) = e^{-\lambda t} \sum_{r=i}^{n} s_{ir}(t) + \sum_{m=1}^{\infty} \frac{(\lambda t)^m}{m!} e^{-\lambda t} \sum_{r=i}^{n} s_{ir}(t) * g_{k-1_{r+1}}^{(m)}(t), \tag{6}$$

where $\mathbf{1}_j = (0, ..., 0, 1, 0, ..., 0)$ with the 1 in the j^{th} position and

$$s_{ir}(t) = (1 - p_i)b_{ii}(t) * ... * (1 - p_{r-1})b_{ir-1}(t) * p_r b_{ir}(t), \qquad r = i, i+1, ..., n,$$

is in fact the total time the P_i retrial customer holds the server from the epoch he finds a position for service until the epoch he joins the K_r^{th} retrial box and becomes a P_{r+1} customer or departs from the system (case $r = n$). By taking LST in (6) above we arrive at

$$D^{(i)}(s, z) = a_i(x(z), z_{i+1}, ..., z_n),$$

where the function $a_i(x(z), z_{i+1}, ..., z_n)$ has been defined in (3).

Define finally $C^{(i)}$ as the time interval from the epoch at which a P_i customer finds a position for service until the epoch the server becomes for the first time idle and ready to accept the next customer from outside or from a retrial box. If $N_j(C^{(i)})$ is the number of new P_j customers during $C^{(i)}$ and define

$$c_{k}^{(i)}(t)dt = P[N_j(C^{(i)}) = k_j \;\; j = 2, 3, ..., n, \;\; t < C^{(i)} \leq t + dt],$$
$$C^{(i)}(s, z) = \sum_{k=0}^{\infty} z^k \int_{t=0}^{\infty} e^{-st} c_{k}^{(i)}(t)dt,$$

then it is easy to realize that

$$C^{(i)}(0, z) = a_i(x(z), z_{i+1}, ..., z_n)v^*(0, z). \tag{7}$$

We have to state here the following theorem. The proof is similar to the proof of Theorem 3.2 in Moutzoukis and Langaris [14] and it is omitted here.

Theorem 2 *For any permutation $(i_2, i_3, ..., i_n)$ of the set $(2, 3, ..., n)$ and for (a) $|z_{i_m}| < 1$ for any specific $m = j+1, ..., n$, and $|z_{i_r}| \leq 1$, for all other $r = j+1, ..., n$ with $r \neq m$, or (b) $|z_{i_r}| \leq 1$, for all $r = j+1, ..., n$, and $\bar{\rho}_{i_{j-1}} > 1$, or (c) $|z_{i_r}| \leq 1$, for all $r = j+1, ..., n$, and $\bar{\rho}_{i_j} > 1 \geq \bar{\rho}_{i_{j-1}}$, the equation*

$$z_{i_j} - C^{(i_j)}(0, \, \boldsymbol{w}_{i_{j-1}}(z_{i_j}, z_{i_{j+1}}, ..., z_{i_n})) = 0, \tag{8}$$

has, for $j = 2, 3, ..., n$, one and only one root, $z_{i_j} = x_{i_j}(z_{i_{j+1}}, ..., z_{i_n})$, $j \neq n$, $z_{i_n} = x_{i_n}$ say, inside the region $|z_{i_j}| < 1$, where the vector $\boldsymbol{w}_{i_j}(z_{i_{j+1}}, z_{i_{j+2}}, ..., z_{i_n})$ is defined by

$$\boldsymbol{w}_{i_1}(z_{i_2}, z_{i_3}, ..., z_{i_n}) = (z_2, z_3, ..., z_n),$$
$$\boldsymbol{w}_{i_2}(z_{i_3}, z_{i_4}, ..., z_{i_n}) = \boldsymbol{w}_{i_1}(x_{i_2}(z_{i_3}, ..., z_{i_n}), z_{i_3}, ..., z_{i_n}),$$
$$\boldsymbol{w}_{i_k}(z_{i_{k+1}}, z_{i_{k+2}}, ..., z_{i_n}) = \boldsymbol{w}_{i_{k-1}}(x_{i_k}(z_{i_{k+1}}, ..., z_{i_n}), z_{i_{k+1}}, ..., z_{i_n}), \quad k = 2, ..., n-1,$$

while $\bar{\rho}_{i_1} \equiv \rho_1$ and

$$\bar{\rho}_{i_j} = \frac{\partial}{\partial z_{i_j}} C^{(i_j)}(0, \, \boldsymbol{w}_{i_{j-1}}(z_{i_j}, z_{i_{j+1}}, ..., z_{i_n}))|_{z_{i_j} = z_{i_{j+1}} = ... = z_{i_n} = 1}.$$

Moreover for real $z_{i_r} = 1$, $r = j+1, ..., n$, and $\bar{\rho}_{i_{j-1}} \leq 1$ the root $x_{i_j}(1, ..., 1)$ is the smallest positive real root of (8) with $x_{i_j}(1, ..., 1) < 1$ if $\bar{\rho}_{i_j} > 1$ and $x_{i_j}(1, ..., 1) = 1$ for $\bar{\rho}_{i_j} \leq 1$.

One can show here that, for any permutatation $(i_2, i_3, ..., i_n)$ of the set $(2, 3, ..., n)$, the last term $\bar{\rho}_{i_n}$ $(> \bar{\rho}_{i_{n-1}} > ... > \bar{\rho}_{i_2})$ is given by,

$$\bar{\rho}_{i_n} = \frac{\rho_{i_n} + \rho_{0i_n} + \delta_{\{i_n < n\}} p_{i_n - 1} \sum_{k=i_n+1}^{n} (\rho_k + \rho_{0k})}{1 - \rho_1 - \sum_{k=2}^{i_n - 1} (\rho_k + \rho_{0k}) - \delta_{\{i_n < n\}} (1 - p_{i_n - 1}) \sum_{k=i_n+1}^{n} (\rho_k + \rho_{0k})}, \tag{9}$$

and so it is clear comparing relations (1), (9) that, for $\rho^* < 1$, $\bar{\rho}_{i_n}$ (and all other $\bar{\rho}_{i_j}$) is always less than one.

4 Steady states analysis

Let us assume that a state of statistical equilibrium exists and let N_i, $i = 1, 2, ..., n$ denote the number of P_i customers in the system. Let also

$$\xi = \begin{cases} 0 & if\ server\ on\ vacation, \\ (i,j) & if\ server\ busy\ on\ j\ phase\ with\ P_i\ customer, \\ id & if\ server\ idle, \end{cases}$$

and

$$\begin{aligned} &q(\boldsymbol{k}) = P(\xi = id,\ N_1 = 0,\ N_m = k_m,\ m = 2, 3, ..., n), \\ &p_0(\bar{\boldsymbol{k}}, x)dx = P(\xi = 0,\ N_m = k_m,\ m = 1, 2, ..., n,\ \ x < \bar{U}_0(t) \le x + dx), \\ &p_{ij}(\bar{\boldsymbol{k}}, x)dx = P(\xi = (i,j),\ N_m = k_m,\ m = 1, 2, ..., n,\ \ x < \bar{U}_{ij}(t) \le x + dx), \end{aligned}$$

where, as it is stated before, $\boldsymbol{k} = (k_2, ..., k_n)$, $\bar{\boldsymbol{k}} = (k_1, k_2, ..., k_n) \equiv (k_1, \boldsymbol{k})$, and $\bar{U}_{ij}(t)$, $\bar{U}_0(t)$ the elapsed service or vacation time respectively. If finally

$$\begin{aligned} &Q(\boldsymbol{z}) = \textstyle\sum_{\boldsymbol{k} \ge \boldsymbol{0}} q(\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}} \equiv \sum_{k_2 \ge 0} \cdots \sum_{k_n \ge 0} q(k_2, ..., k_n) z_2^{k_2} z_3^{k_3} ... z_n^{k_n}, \\ &P_0(\bar{\boldsymbol{z}}, x) = \textstyle\sum_{\bar{\boldsymbol{k}} \ge \bar{\boldsymbol{0}}} p_0(\bar{\boldsymbol{k}}, x) \bar{\boldsymbol{z}}^{\bar{\boldsymbol{k}}}, \\ &P_{ij}(\bar{\boldsymbol{z}}, x) = \textstyle\sum_{\bar{\boldsymbol{k}} \ge \bar{\boldsymbol{0}}} p_{ij}(\bar{\boldsymbol{k}}, x) \bar{\boldsymbol{z}}^{\bar{\boldsymbol{k}}}, \qquad i, j = 1, 2, \end{aligned}$$

then we arrive easily, for $x > 0$, at

$$\begin{aligned} P_0(\bar{\boldsymbol{z}}, x) &= P_0(\bar{\boldsymbol{z}}, 0)(1 - B_0(x)) \exp[-(\lambda - \lambda z_1)x], \\ P_{ij}(\bar{\boldsymbol{z}}, x) &= P_{ij}(\bar{\boldsymbol{z}}, 0)(1 - B_{ij}(x)) \exp[-(\lambda - \lambda z_1)x], \end{aligned} \tag{10}$$

and

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(\boldsymbol{z}) + \lambda Q(\boldsymbol{z}) = P_0((0, \boldsymbol{z}), 0) \beta_0^*(\lambda), \tag{11}$$

with $\beta_0^*(.)$ the LST of $b_0(.)$. For the boundary conditions ($x=0$) we obtain in a similar way

$$\begin{aligned} P_0((0,\boldsymbol{z}),0) &= \sum_{m=1}^{n-1} p_m z_{m+1} \sum_{i=1}^{m} P_{im}((0,\boldsymbol{z}),0)\beta_{im}^*(\lambda) \\ &\quad + \sum_{i=1}^{n} P_{in}((0,\boldsymbol{z}),0)\beta_{in}^*(\lambda), \end{aligned} \tag{12}$$

$$\begin{aligned} &P_{jj}((0,\boldsymbol{z}),0) = \mu_j \tfrac{d}{dz_j} Q\ (\boldsymbol{z}), && j=2,...,n, \\ &P_{1j}(\bar{\boldsymbol{z}},0) = P_{11}(\bar{\boldsymbol{z}},0)\textstyle\prod_{m=1}^{j-1}(1-p_m)\beta_{1m}^*(\lambda-\lambda z_1), && j=2,...,n, \\ &P_{ij}(\bar{\boldsymbol{z}},0) = \ \mu_i \tfrac{d}{dz_i} Q(\boldsymbol{z})\textstyle\prod_{m=i}^{j-1}(1-p_m)\beta_{im}^*(\lambda-\lambda z_1), && \begin{matrix} i=2,...,n \\ j=i+1,...,n, \end{matrix} \end{aligned} \tag{13}$$

while for the $P_{11}(\bar{\boldsymbol{z}},0)$ we obtain using relations (12) and (13)

$$\begin{aligned} P_{11}(\bar{\boldsymbol{z}},0) &= \{\lambda z_1 Q(\boldsymbol{z}) + \sum_{j=2}^{n} a_j(z_1, z_{j+1}, ..., z_n) P_{jj}((0,\boldsymbol{z}),0) \\ &\quad - P_0((0,\boldsymbol{z}),0)[1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda z_1)]\}/[z_1 - a_1(\bar{\boldsymbol{z}})]. \end{aligned} \tag{14}$$

Replacing now in the numerator of (14) the zero $x(\boldsymbol{z})$ of the denominator, we arrive at

$$P_0((0,\boldsymbol{z}),0) = \frac{\lambda x(\boldsymbol{z})Q(\boldsymbol{z}) + \sum_{j=2}^{n} a_j(x(\boldsymbol{z}), z_{j+1},...,z_n)P_{jj}((0,\boldsymbol{z}),0)}{1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda x(\boldsymbol{z}))}, \tag{15}$$

and substituting back from (12)-(15) in (10) and (11) and integrating with respect to x we obtain for $j=2,...,n$

$$e_{1j}(z_1)P_{1j}(\bar{\boldsymbol{z}}) = \ e_{11}(z_1)P_{11}(\bar{\boldsymbol{z}}) \prod_{m=1}^{j-1}(1-p_m)\beta_{1m}^*(\lambda-\lambda z_1), \tag{16}$$

$$e_{jj}(z_1)P_{jj}(\bar{\boldsymbol{z}}) = \mu_j \frac{d}{dz_j} Q(\boldsymbol{z}), \tag{17}$$

$$e_{ij}(z_1)P_{ij}(\bar{\boldsymbol{z}}) = \ \mu_i \frac{d}{dz_i} Q(\boldsymbol{z}) \prod_{m=i}^{j-1}(1-p_m)\beta_{im}^*(\lambda-\lambda z_1), \quad \begin{matrix} i=2,...,n \\ j=i+1,...,n, \end{matrix} \tag{18}$$

$$e_0(z_1)P_0(\bar{\boldsymbol{z}}) = \ \frac{\lambda x(z_2,...,z_n)Q(\boldsymbol{z}) + \sum_{j=2}^{n} e_{jj}(z_1)a_j(x(z_2,...,z_n), z_{j+1},...,z_n)P_{jj}(\bar{\boldsymbol{z}})}{1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda x(\boldsymbol{z}))}, \tag{19}$$

$$\begin{aligned} e_{11}(z_1)P_{11}(\bar{\boldsymbol{z}}) \quad &= \{\lambda[z_1 - x(\boldsymbol{z})]Q(\boldsymbol{z}) + \textstyle\sum_{j=2}^{n} e_{jj}(z_1)[a_j(z_1, z_{j+1},...,z_n) \\ &\quad - a_j(x(\boldsymbol{z}), z_{j+1},...,z_n)]P_{jj}(\bar{\boldsymbol{z}}) + \ e_0(z_1)[\beta_0^*(\lambda-\lambda z_1) \\ &\quad - \beta_0^*(\lambda-\lambda x(\boldsymbol{z}))]P_0(\bar{\boldsymbol{z}})\}/[z_1 - a_1(\bar{\boldsymbol{z}})], \end{aligned} \tag{20}$$

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(z) + \lambda Q(z) = e_0(z_1) P_0(\bar{z}) \beta_0^*(\lambda), \tag{21}$$

where in general $P.(\bar{z}) = \int P.(\bar{z}, x)dx$ and

$$e_{ij}(z_1) = \frac{\lambda - \lambda z_1}{1 - \beta_{ij}^*(\lambda - \lambda z_1)}, \qquad e_{ij}(1) = \frac{1}{\bar{b}_{ij}}.$$

We will use in the sequel the expressions above to obtain all generating functions at the point $\bar{z} = \bar{\mathbf{1}}$.

Theorem 3 *For $\rho^* < 1$ the generating functions $P_{ij}(.)$, $P_0(.)$, $Q(.)$ at the point $\bar{z} = \bar{\mathbf{1}}$ are given by*

$$\begin{aligned}
&P_{11}(\bar{\mathbf{1}}) = \lambda \bar{b}_{11}, \quad P_{jj}(\bar{\mathbf{1}}) = \lambda p_{j-1} \bar{b}_{jj}, && j = 2, ..., n \\
&P_{ij}(\bar{\mathbf{1}}) = \lambda p_{i-1} \bar{b}_{ij} \textstyle\prod_{m=i}^{j-1} (1 - p_m), && i = 1, ..., n \\
& && j = i+1, ..., n \\
&Q(\mathbf{1}) = \frac{1-\rho^*}{1+\lambda \bar{b}_0/\beta_0^*(\lambda)}, \\
&P_0(\bar{\mathbf{1}}) = \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)} [\textstyle\sum_{j=2}^{n} p_{j-1} + Q(\mathbf{1})].
\end{aligned} \tag{22}$$

Proof: Let us define

$$\mathcal{N}(\bar{z}) = Q(z) + P_0(\bar{z}) + \sum_{i=1}^{n} \sum_{j=i}^{n} P_{ij}(\bar{z}), \tag{23}$$

then from relations (16), (17) and (18)

$$P_{ij}(\bar{z}) = \frac{e_{ii}(z_1)}{e_{ij}(z_1)} P_{ii}(\bar{z}) \prod_{m=i}^{j-1} (1 - p_m) \beta_{im}^*(\lambda - \lambda z_1), \qquad \begin{array}{l} i = 1, 2, ..., n \\ j = i+1, ..., n \end{array}$$

i.e.

$$P_{ij}(\bar{\mathbf{1}}) = P_{ii}(\bar{\mathbf{1}}) \frac{\bar{b}_{ij}}{\bar{b}_{ii}} \prod_{m=i}^{j-1} (1 - p_m), \qquad \begin{array}{l} i = 1, 2, ..., n \\ j = i+1, ..., n \end{array} \tag{24}$$

and substituting back in (23), using (19) and observing that $\mathcal{N}(\bar{\mathbf{1}}) = \bar{\mathbf{1}}$ we obtain after manipulations

$$\begin{aligned}
&\tfrac{\rho_1}{\lambda \bar{b}_{11}} P_{11}(\bar{\mathbf{1}}) + (1 + \tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) Q(\mathbf{1}) \\
&\qquad + \textstyle\sum_{j=2}^{n} \frac{P_{jj}(\bar{\mathbf{1}})}{\bar{b}_{jj}} [\bar{b}_{jj} + \sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1} (1 - p_m) + \frac{\bar{b}_0}{\beta_0^*(\lambda)}] = 1.
\end{aligned} \tag{25}$$

Using now (19) in (20) and putting $\bar{z} = \bar{\mathbf{1}}$ we arrive at

$$\begin{aligned}
\frac{P_{11}(\bar{\mathbf{1}})}{\bar{b}_{11}} &= \frac{\lambda}{1 - \rho_1} \{(1 + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) Q(\mathbf{1}) \\
&\quad + \sum_{j=2}^{n} \frac{P_{jj}(\bar{\mathbf{1}})}{\bar{b}_{jj}} [\bar{b}_{jj} + \sum_{k=j+1}^{n} \bar{b}_{jk} \prod_{m=j}^{k-1} (1 - p_m) + \frac{\bar{b}_0}{\beta_0^*(\lambda)}]\},
\end{aligned} \tag{26}$$

where ρ_1 has been defined in Lemma 1, and so from (24)-(26) above we conclude

$$P_{11}(\bar{1}) = \lambda\bar{b}_{11}, \qquad P_{1j}(\bar{1}) = \lambda\bar{b}_{1j}\prod_{m=1}^{j-1}(1-p_m), \qquad j = 2,3,...,n. \tag{27}$$

Multiplying relation (17) by z_j, adding for all j, subtracting from (21) and using (19) we arrive at

$$\frac{\sum_{j=2}^{n} e_{jj}(z_1)P_{jj}(\bar{z})}{Q(z)}[D_j(\bar{z}) - z_j] = \lambda[1 - x(z)f(x(z))], \tag{28}$$

with

$$f(x(z)) = \frac{\beta_0^*(\lambda)}{1 + \beta_0^*(\lambda) - \beta_0^*(\lambda - \lambda x(z))}, \tag{29}$$

$$D_j(z) = f(x(z))a_j(x(z), z_{j+1}, ..., z_n). \tag{30}$$

Now it is clear from (5), (7) and (29), (30) that

$$D_i(z) \equiv C^{(i)}(0, z),$$

and so for $\rho^* < 1$ Theorem 2 holds for $D_i(z)$.

By putting now $z_1 = 1$ and, for any permutation $(i_2, i_3, ..., i_n)$ of $(2, 3, ..., n)$, replacing z_{i_k} by the corresponding zero $x_{i_k}(z_{i_{k+1}}, ..., z_{i_n})$ we succeed to eliminate all except one terms in the left hand part of (28) and arrive at

$$P_{jj}(\bar{1}) = \frac{\lambda p_{j-1}\bar{b}_{jj}}{1-\rho^*}(1 + \frac{\lambda\bar{b}_0}{\beta_0^*(\lambda)})Q(1), \qquad j = 2,3,...,n, \tag{31}$$

and replacing in (25) we obtain after manipulations

$$Q(1) = \frac{1-\rho^*}{1 + \frac{\lambda\bar{b}_0}{\beta_0^*(\lambda)}}, \tag{32}$$

which is the third of (22).

From (32) and (31) we obtain the first of (22) and putting back $P_{jj}(\bar{1})$ in (24) and (19) we arrive at the second and forth of (22) respectively and the theorem has been proved. □

5 Mean number of ordinary and retrial customers

For $p_0 \equiv 1$ and $j = 1, ..., n$, let us define now

$$\begin{aligned}\pi_j = \; & p_{j-1}\{\bar{b}_{jj}^{(2)} + \sum_{k=j+1}^{n}\bar{b}_{jk}^{(2)}\prod_{m=j}^{k-1}(1-p_m) \\ & +2\sum_{r=j}^{n-1}\prod_{m=j}^{r}(1-p_m)\bar{b}_{jr}[\bar{b}_{jr+1} + \sum_{k=r+2}^{n}\bar{b}_{jk}\prod_{m=r+1}^{k-1}(1-p_m)]\},\end{aligned} \tag{33}$$

then, for the mean length of the ordinary queue,

Theorem 4 *The mean number of P_1 customers, in the ordinary queue is given by,*

$$E(N_1) = \frac{\lambda^2}{2(1-\rho_1)}\{\frac{\bar{b}_0}{\beta_0^*(\lambda)}[\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \sum_{k=1}^{n}\pi_k\}, \tag{34}$$

where $Q(\mathbf{1})$ and π_k are given by (32), (33) respectively.

Proof: Differentiating relations (17), (18), (19), with respect to z_1, and setting $z_l = 1,\ l = 1, 2, ..., n$, we arrive easily at,

$$\begin{aligned}
\frac{\partial P_{ii}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2 p_{i-1}\bar{b}_{ii}^{(2)}}{2}, \\
\frac{\partial P_{ij}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2 p_{i-1}\bar{b}_{ij}^{(2)}}{2}\prod_{m=i}^{j-1}(1-p_m) + \lambda^2 p_{i-1}\bar{b}_{ij}\prod_{m=i}^{j-1}(1-p_m)\sum_{m=i}^{j-1}\bar{b}_{im}, \\
& \qquad i = 2, 3, ..., n,\ j = i+1, ..., n, \\
\frac{\partial P_0(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^2\bar{b}_0^{(2)}}{2\beta_0^*(\lambda)}[\textstyle\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})].
\end{aligned} \tag{35}$$

In a similar way, from (20), (16),

$$\begin{aligned}
\frac{\partial P_{11}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^3\bar{b}_{11}}{2(1-\rho_1)}[\frac{\bar{b}_0^{(2)}}{\beta_0^*(\lambda)}[\textstyle\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \displaystyle\sum_{k=1}^{n}\pi_k] + \frac{\lambda^2\bar{b}_{11}^{(2)}}{2}, \\
\frac{\partial P_{1j}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} &= \frac{\lambda^3\bar{b}_{1j}}{2(1-\rho_1)}\prod_{m=1}^{j-1}(1-p_m)[\frac{\bar{b}_0^{(2)}}{\beta_0^*(\lambda)}[\textstyle\sum_{k=2}^{n} p_{k-1} + Q(\mathbf{1})] + \displaystyle\sum_{k=1}^{n}\pi_k] \\
& \quad + \lambda^2\prod_{m=1}^{j-1}(1-p_m)[\bar{b}_{1j}\sum_{m=1}^{j-1}\bar{b}_{1m} + \frac{\bar{b}_{1j}^{(2)}}{2}], \qquad j = 2, 3, ..., n.
\end{aligned} \tag{36}$$

Observing now that

$$E(N_1) = \frac{\partial\mathcal{N}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} = \sum_{i=1}^{n}\sum_{j=i}^{n}\frac{\partial P_{ij}(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}} + \frac{\partial P_0(\bar{z})}{\partial z_1}|_{\bar{z}=\bar{1}},$$

and replacing from (35) and (36) we obtain relation (34) and the theorem has been proved. □

Before giving the mean queue lengths for the retrial customers we have to state some preliminary results. Let, for $k, j = 2, ..., n,\ m = 1, 2,$

$$h_k^{(m)} = \frac{d^m x(z)}{dz_k^m}|_{z=1}, \qquad \hat{\rho}_{jk}^{(m)} = \frac{d^m\alpha_j(x(z), ..., z_k, ...1)}{dz_k^m}|_{z=1},$$

where $x(z)$ is defined in Lemma 1 (note that $\hat{\rho}_{jk}^{(u)} \neq \hat{\rho}_{kj}^{(u)},\ k \neq j$). Then after some algebra we obtain,

$$\begin{aligned}
h_k^{(1)} &= \tfrac{p_{k-1}}{(1-\rho_1)} \prod_{m=1}^{k-2}(1-p_m), \\
h_k^{(2)} &= \tfrac{\lambda^2 (h_k^{(1)})^2}{(1-\rho_1)}\pi_1 + 2\lambda (h_k^{(1)})^2 \sum_{m=1}^{k-1} \bar{b}_{1m}, \\
\hat{\rho}_{jk}^{(1)} &= \lambda h_k^{(1)}(\bar{b}_{jj} + \textstyle\sum_{r=j+1}^{n} \bar{b}_{jr} \prod_{m=j}^{r-1}(1-p_m)) + \delta_{\{k>j\}}[p_{k-1}\prod_{m=j}^{k-2}(1-p_m)], \\
\hat{\rho}_{jk}^{(2)} &= \lambda h_k^{(2)}(\bar{b}_{jj} + \textstyle\sum_{r=j+1}^{n} \bar{b}_{jr} \prod_{m=j}^{r-1}(1-p_m)) + \tfrac{(\lambda h_k^{(1)})^2}{p_{j-1}}\pi_j \\
&\quad + \delta_{\{k>j\}} 2\lambda h_k^{(1)} p_{k-1} \prod_{m=j}^{k-2}(1-p_m) \sum_{r=j}^{k-1} \bar{b}_{jr}, \qquad k,j = 2,...,n.
\end{aligned}$$

with $\prod_{m=j}^{i}(1-p_m) = 1$ for $j > i$. Moreover define

$$\begin{aligned}
s_k = &\ \tfrac{\lambda Q(1,...,1) h_k^{(2)}}{2p_{k-1}\prod_{m=1}^{k-2}(1-p_m)} + \tfrac{\lambda^2 p_{k-1}}{\mu_k(1-\rho_1)} + \tfrac{P_0(1,...,1)}{2p_{k-1}\bar{b}_0 \prod_{m=1}^{k-2}(1-p_m)}[\lambda \bar{b}_0 h_k^{(2)} + (\lambda h_k^{(1)})^2 \bar{b}_0^{(2)}] \\
&+ \tfrac{\lambda}{2p_{k-1}\prod_{m=1}^{k-2}(1-p_m)} \sum_{j=2}^{n} p_{j-1}\hat{\rho}_{jk}^{(2)} + \tfrac{\lambda^2 \bar{b}_0}{(1-\rho_1)\beta_0^*(\lambda)}[(1-\rho^*)h_k^{(1)} + \tfrac{\lambda p_{k-1}}{\mu_k}] \\
&+ \tfrac{\lambda}{(1-\rho_1)} \sum_{j=2}^{n} \rho_{0j}(\hat{\rho}_{jk}^{(1)} + \tfrac{\lambda h_k^{(1)} \bar{b}_0}{\beta_0^*(\lambda)}), \qquad k = 2,...,n.
\end{aligned}$$

and denote

$$D_{kl} = \frac{\partial^2}{\partial z_k \partial z_l} Q(z)|_{z=1}, \qquad k,l = 2,3,...,n.$$

Note here that $D_{kl} = D_{lk}\ \forall\ k,l$. Then we state the following theorem.

Theorem 5 *The quantities $D_{kl},\ k,\ l = 2,...,n$ can be found as the solution of the system of linear equations,*

$$\begin{aligned}
(\mu_k + \mu_l)D_{kl} = &\sum_{j=2}^{n-1} \mu_j D_{kj}(p_{l-1}\prod_{m=j}^{l-2}(1-p_m))\delta_{\{l>j\}} + \sum_{j=2}^{n-1} \mu_j D_{lj}\delta_{\{k>j\}} \\
&\times (p_{k-1}\prod_{m=j}^{k-2}(1-p_m)) + p_{l-1}\prod_{m=1}^{l-2}(1-p_m)[\sum_{j=2}^{n} \tfrac{\mu_j(\rho_j+\rho_{0j})}{p_{j-1}(1-\rho_1)} D_{kj} + s_k] \\
&+ p_{k-1}\prod_{m=1}^{k-2}(1-p_m)[\sum_{j=2}^{n} \tfrac{\mu_j(\rho_j+\rho_{0j})}{p_{j-1}(1-\rho_1)} D_{lj} + s_l].
\end{aligned} \qquad (37)$$

Proof: Replacing (19) to (20) we arrive at

$$e_0(z_1)P_0(\bar{z}) = \frac{\lambda z_1 Q(z) + \sum_{j=2}^{n} e_{jj}(z_1)a_j(z_1,z_{j+1},...,z_n)P_{jj}(\bar{z}) + e_{11}(z_1)P_{11}(\bar{z})(a_1(\bar{z})-z_1)}{1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda z_1)},$$

and replacing to (21) and setting $z_1 = 1$ we obtain

$$\sum_{j=2}^{n} \mu_j z_j \frac{\partial}{\partial z_j} Q(z) = \sum_{j=2}^{n-1} \frac{P_{jj}(1,z)}{\bar{b}_{jj}} a_j(1, z_{j+1}, ..., z_n) + \frac{P_{11}(1,z)}{\bar{b}_{11}}(a_1(1,z) - 1). \quad (38)$$

Now adding relation (17) for all $j = 2, ..., n$, putting $z_1 = 1$ and subtracting from (38) we arrive at our basic equation,

$$\sum_{j=2}^{n} \mu_j (z_j - 1) \frac{\partial}{\partial z_j} Q(z) = \sum_{j=1}^{n-1} \frac{P_{jj}(1,z)}{\bar{b}_{jj}} [a_j(1, z_{j+1}, ..., z_n) - 1]. \quad (39)$$

Taking finally derivatives above with respect to z_k, z_l, and using the fact that from (17) $\frac{\partial P_{jj}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = \mu_j D_{jk} \bar{b}_{jj}$, $\forall\ k,\ j = 2, 3, ..., n$, we arrive after some algebra at relation (37). □

Now we are ready to give the mean number of customers in the retrial boxes.

Lemma 6 *The mean number of P_k, $k = 2, 3, ..., n$ customers in K_{k-1}^{th} retrial box is given by*

$$\begin{aligned} E(N_k) = & \sum_{m=2}^{n} \frac{\mu_m(\rho_m + \rho_{0m})}{\lambda p_{m-1}(1-\rho_1)} D_{mk} + \frac{s_k \rho_1}{\lambda} + \sum_{m=2}^{n} \rho_{0m}(\hat{\rho}_{mk}^{(1)} + \frac{\lambda h_k^{(1)} \bar{b}_0}{\beta_0^*(\lambda)}) \\ & + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}[(1-\rho^*)h_k^{(1)} + \frac{\lambda p_{k-1}}{\mu k}] + \frac{\lambda p_{k-1}}{\mu_k}. \end{aligned} \quad (40)$$

Proof: Differentiating (20), (16), with respect to z_k, and setting $z_l = 1$, $l = 1, 2, ..., n$, we obtain after manipulations,

$$\begin{aligned} \frac{\partial P_{11}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \bar{b}_{11}[\sum_{m=2}^{n} \frac{\mu_m(\rho_m + \rho_{0m})}{p_{m-1}(1-\rho_1)} D_{mk} + s_k], \\ \frac{\partial P_{1j}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \frac{\bar{b}_{1j} \prod_{r=1}^{j-1}(1-p_r)}{\bar{b}_{11}} \frac{\partial P_{11}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}}, \quad j = 2, ..., n. \end{aligned} \quad (41)$$

In a similar way from (17), (18), (19),

$$\begin{aligned} \frac{\partial P_{ii}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \mu_i D_{ik} \bar{b}_{ii}, \\ \frac{\partial P_{ij}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \frac{\bar{b}_{ij} \prod_{r=i}^{j-1}(1-p_r)}{\bar{b}_{ii}} \frac{\partial P_{ii}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}}, \quad \begin{array}{l} i = 2, ..., n, \\ j = i+1, ..., n, \end{array} \\ \frac{\partial P_0(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = & \sum_{m=2}^{n} \frac{\mu_m \rho_{0m}}{\lambda p_{m-1}} D_{mk} + \frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}[(1-\rho^*)h_k^{(1)} + \frac{\lambda p_{k-1}}{\mu_k}] \\ & + \sum_{m=2}^{n} \rho_{0m}(\hat{\rho}_{mk}^{(1)} + \frac{\lambda h_k^{(1)} \bar{b}_0}{\beta_0^*(\lambda)}). \end{aligned} \quad (42)$$

Observing now that, for any $k = 2, 3, ..., n$,

$$E(N_k) = \frac{\partial \mathcal{N}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} = \frac{\partial Q(z)}{\partial z_k}|_{z=1} + \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{\partial P_{ij}(\bar{z})}{\partial z_k}|_{\bar{z}=\bar{1}} + \frac{\partial P_0(\bar{z})}{z_k}|_{\bar{z}=\bar{1}}, \quad (43)$$

and replacing from (41), (42) to (43) we arrive easily at (40) and the theorem has been proved. □

6 Numerical Results

In this section we consider a system of $n = 4$ phases of service and so three retrial boxes and use the formulae derived previously to obtain numerical results and to investigate the way the mean number of customers in the retrial boxes $E(N_i)$ $i = 2, 3, 4$ are affected when we vary the mean vacation time $\bar{b}_0$, the mean service time of the ordinary customers $\bar{b}_{11}$, and the mean retrial interval in the first box $E(retrial\ K_1) = 1/\mu_1$, always for increasing values of the mean arrival rate λ.

To construct the tables we assume that the vacation time U_0 and the service times follow exponential distributions with p.d.f.'s respectively,

$$b_0(x) = \frac{1}{\bar{b}_0} e^{-(1/\bar{b}_0)x}, \qquad b_{ij}(x) = \frac{1}{\bar{b}_{ij}} e^{-(1/\bar{b}_{ij})x}, \qquad \begin{array}{l} i = 1, ..., 4 \\ j = i, ..., 4. \end{array}$$

Moreover we assume that in all tables below $\bar{b}_{14} = 0.2$, $\bar{b}_{12} = \bar{b}_{22} = \bar{b}_{23} = \bar{b}_{33} = 0.33$, $\bar{b}_{13} = \bar{b}_{24} = \bar{b}_{34} = \bar{b}_{44} = 0.25, p_1 = 0.7$, $p_2 = 0.5$, $p_3 = 0.1$. Finally $\mu_2 = 0.5$, $\mu_3 = 2$.

Table 1 shows the way $E(N_i)$ $i = 2, 3, 4$ changes when we vary the mean vacation time, for increasing values of the mean arrival rate λ. Here one can observe the crucial role that the vacation plays on the number of retrial customers. Thus, even for a small value of λ, $\lambda = 0.15$ for example, $E(N_2)$ increases from 0.1488 to 46.026 when we pass from a system without vacation period ($\bar{b}_0 = 0$) to the system with $\bar{b}_0 = 2.7$, while the corresponding value for $E(N_3)$ increases from 0.2029 to 64.83. When now the arrival rate λ becomes $\lambda = 0.42$ then even a small change from $\bar{b}_0 = 0$ to $\bar{b}_0 = 0.6$ increases dramatically the mean number of retrial customers to 267.02 in retrial box K_1 and to 380.45 in retrial box K_2 respectively. Thus we must be very careful on the vacation period that we must allow, to avoid overcrowded retrial boxes. The behavior of the third retrial box K_3 ($E(N_4)$) is smoother, and it shows us a way to reduce this dramatic effect of the vacation period by allowing faster retrials ($E(retrial\ K_3) = 1/\mu_3 = 0.5$

here) and/or less preference of the box ($p_3 = 0.1$).

$\lambda \backslash \bar{b}_0$		0	0.2	0.6	1.3	2.7
0.15	$E(N_2)$	0.1488	0.1926	0.3084	0.6816	46.026
	$E(N_3)$	0.2029	0.2451	0.3602	0.7605	64.83
	$E(N_4)$	0.0112	0.0161	0.0282	0.0623	3.316
0.27		0.3798	0.5273	1.1548	120.04	
		0.4958	0.6693	1.3906	170.57	
		0.0287	0.0435	0.0954	8.599	
0.42		1.045	1.8373	267.02		
		1.3022	2.2913	380.45		
		0.0768	0.1395	19.077		
0.59		4.1926	221.4			
		5.2556	314.4			
		0.2956	15.814			
0.71		126.8				
		179.3				
		9.0327				

Table 1 : *Values of* $E(N_i)$, $i = 2, 3, 4$, *for* $\mu_1 = 1$, $\bar{b}_{11} = 0.5$.

Similar observations can be deduced from Table 2 that contains values of $E(N_i)$ $i = 2, 3, 4$ when we vary the mean first stage service $\bar{b}_{11}$. One can observe again the way the mean number of retrial customers in each box increases when $\bar{b}_{11}$ increases. An increase that depends on how fast or slow the mean retrial $E(retrial\ K_i)$ is and/or on the preference that customers show to the corresponding box p_i.

$\lambda \backslash \bar{b}_{11}$		0.2	0.8	1.3	2.1	2.8	5.5
0.15	$E(N_2)$	0.1711	0.2233	0.3014	0.5481	1.0297	18522.3
	$E(N_3)$	0.2257	0.271	0.3291	0.4882	0.7737	26410.6
	$E(N_4)$	0.0145	0.0181	0.0226	0.0346	0.0595	1321.15
0.25		0.3646	0.6088	1.1043	4.3585	193.69	
		0.4783	0.7108	1.1131	3.97	263.19	
		0.0306	0.047	0.0755	0.25	13.378	
0.3		0.5051	0.9877	2.267	64.9		
		0.662	1.1428	2.354	85.09		
		0.042	0.0742	0.1496	4.423		
0.4		0.9447	3.034	112.03			
		1.2382	3.6085	156.15			
		0.0765	0.2161	7.9094			
0.5		1.8448	108.39				
		2.4349	152.81				
		0.1444	7.717				
0.71		92.598					
		131.62					
		6.632					

Table 2 : *Values of* $E(N_i)$, $i = 2, 3, 4$, *for* $\mu_1 = 1$, $\bar{b}_0 = 0.2$.

Table 3 finally depicts the way the $E(N_i)$ $i = 2, 3, 4$ are affected when we vary the mean retrial rate in the first box $E(retrial) = 1/\mu_1$. One can observe here not only the increase of $E(N_2)$, but mainly the reduction of $E(N_3)$ and $E(N_4)$ in the other two boxes when we increase $1/\mu_1$, a reduction which is more apparent when λ increases.

$\lambda \backslash E(retrial\ K_1)$		0.02	0.2	1	2	10
0.15	$E(N_2)$	0.0452	0.0727	0.1926	0.3397	1.4994
	$E(N_3)$	0.2539	0.2515	0.2454	0.2409	0.2329
	$E(N_4)$	0.0177	0.0171	0.0161	0.0157	0.0151
0.27		0.1423	0.2185	0.5373	0.9183	3.862
		0.7229	0.7087	0.6693	0.6443	0.5987
		0.0521	0.0488	0.0435	0.413	0.0382
0.42		0.4837	0.761	1.8373	3.045	11.953
		2.7032	2.5895	2.2913	2.1153	1.8179
		0.1904	0.1699	0.1395	0.1271	0.1093
0.59		8.33	60.25	221.4	371.7	1347.5
		464.3	419.03	314.4	265.23	192.75
		23.5	21.12	15.814	13.31	9.6763

Table 3 : *Values of* $E(N_i)$, $i = 2, 3, 4$ *for* $\bar{b}_0 = 0.2$, $\bar{b}_{11} = 0.5$.

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